Swimming of a waving plate

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The purpose of this paper is to study the basic principle of fish propulsion. As a simplified model, the two-dimensional potential flow over a waving plate of finite chord is treated. The solid plate, assumed to be flexible and thin, is capable of performing the motion which consists of a progressing wave of given wave-length and phase velocity along the chord, the envelope of the wave train being an arbitrary function of the distance from the leading edge. The problem is solved by applying the general theory for oscillating deformable airfoils. The thrust, power required, and the energy imparted to the wake are calculated, and the propulsive efficiency is also evaluated. As a numerical example, the waving motion with linearly varying amplitude is carried out in detail. Finally, the basic mechanism of swimming is elucidated by applying the principle of action and reaction.

1. Introduction

'How does a fish swim'? is indeed a fascinating question. Geometrically, fishes of many varieties have their bodies in a planar form of finite aspect ratio, that is, their bodies have finite length and width and are thin in the third dimension; other varieties have slender cylindrical forms. Almost without exception all fishes have a flexible body to the extent that they can perform undulating motions in swimming. Physical observations further indicate that in general the motion of the deformable body consists of a train of waves, which are not standing waves, but which progress astern. Furthermore, the wave amplitude usually grows toward the tail. Most fishes propel themselves in water at speeds corresponding to a Reynolds number, based on their length, of the order of 10^5 or higher. Presumably the inertia forces in the surrounding fluid are an important factor in producing the propulsion; the viscosity of the fluid, other than generating drag, is important in creating circulation around the swimming body, and only in this role does it affect the propulsive forces. Taylor (1951, 1952*a*, *b*) has discussed certain aspects of propulsion of swimming bodies. In the first two references Taylor treated the problem of swimming microscopic organisms in viscous fluid where viscosity plays the leading role. In the third reference he investigated the swimming of long narrow animals. It is the purpose of this paper to study the mechanism of marine propulsion for very large Reynolds number and to discuss its important features such as the effect, on the propulsive force and the swimming efficiency, of the wavelength, the phase velocity of the wave, and the spatial variation in amplitude of the wave.

In order to simplify the determination of these complicated effects, this analysis is limited to the relatively simple case of a two-dimensional waving plate of negligible thickness. The fluid is assumed to be incompressible and inviscid, but with the Kutta condition imposed at the trailing edge of the plate. The problem then becomes the general problem of airfoils in unsteady motion. In order to represent the body profile in swimming, the plate must have infinite degrees of freedom. The solution of the case for airfoils in oscillatory motion of general form has been obtained by Küssner & Schwarz (1940); this solution is used as the starting point for further analysis in this work. However, the calculation of the propulsive force on the airfoil in unsteady motion has only been investigated for a limited number of simple cases, presumably because it is of small importance in any application in aeronautics. Kármán & Burgers (1943, p. 304) calculated the propulsive force and the work expended in maintaining the motion, but only for the most simple case of a rigid plate in transverse oscillation, thus providing a qualitative theory for flapping flight. Another calculation of the propulsive force, for the particular case of a rigid wing, is given by Nekrasov (1948). Since the problem is of primary interest in connexion with the flight of birds, and especially with the swimming of fishes, the thrust, the power required for maintaining the motion and the energy imparted to the fluid are calculated here for the general case of a waving deformable plate.

For further application of the general result, consideration is given to the motion of a flexible plate which consists of a train of progressive waves, the amplitude of the wave being taken as an arbitrary function of the distance along the plate, which is assumed to possess a Taylor expansion. In this general case it is found that the time-average of the energy imparted to the fluid, $\langle E_w \rangle$, is always positive; it vanishes if, and only if, the circulation around the swimming plate remains constant. A subsidiary result indicates that the time-average of the power input, $\langle P \rangle$, can be negative (implying that energy can be extracted from the fluid), but only in case of an oscillatory motion with at least two degrees of freedom. It is also shown that when energy is taken out of the fluid, the average thrust on the body cannot be positive. As a simple example, the waving motion with linearly varying amplitude is carried out in detail, the results plotted in diagrams and their physical significance discussed.

Finally, an attempt is made to elucidate the basic mechanism of swimming by applying the principle of action and reaction. When the plate acquires a forward momentum as it swims through a fluid, the fluid is pushed in the backward direction with the net total momentum equal and opposite to that of the action. Investigation of the strength of the vortex sheet shed from the trailing edge of the plate indicates that the forward thrust is positive only when the vorticity is so oriented that the fluid in the wake is pushed backwards from the tail, and the thrust is negative otherwise. Thus the momentum of the fluid in reaction is well concentrated in the vortex wake, appearing in the form of a jet of fluid expelled from the plate.

After this paper had been submitted, the author learned of M. J. Lighthill's work on a related subject (the swimming of slender fish) presented briefly in the Forty-Eighth Wilbur Wright Memorial Lecture (Lighthill 1960a) and subsequently in a separate paper (Lighthill 1960*b*). In these references the theory is worked out for slender fish. The problem dealt with in this paper, however, is concerned with what may be called a two-dimensional flat fish. The theory for swimming of real fishes ought perhaps to lie somewhere in between these two limiting cases. It may be speculated that this subject also has some application in the aeroelasticity of oscillating wings.

2. Formulation of the problem; the acceleration potential

Consider the two-dimensional incompressible flow of an inviscid fluid generated by the motion of a deformable solid plate of zero thickness spanning from $x = -1$ to $x = 1$ in an otherwise uniform stream of constant velocity U in the positive x-direction. The motion of the flexible plate may be prescribed in the general form by

$$y = h(x,t) \quad \text{for} \quad -1 < x < 1, \tag{1}$$

where $h(x,t)$ is an arbitrary continuous function of x for every time t, the maximum amplitude of h and $\partial h/\partial x$ being assumed very small compared with unity. The flow velocity $\mathbf{q}$ has x- and y-components given by $\mathbf{q} = (U+u, v)$ and satisfies the continuity equation

$$\operatorname{div}\mathbf{q} = u_x + v_y = 0. \tag{2}$$

(The subscript x, or y, denotes partial differentiation with respect to x, or y.) By assuming the perturbation velocity u, v to be small compared with U, the Euler equations of motion for the ideal fluid may be linearized to give

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\mathbf{q} = -\frac{1}{\rho}\operatorname{grad} p = \operatorname{grad}\phi, \tag{3}$$

where

$$\phi(x,y,t) = (p_\infty - p)/\rho, \tag{4}$$

in which ρ is the density, p the pressure of the fluid and p_∞ the pressure at infinity. The constant term p_∞ is included in the above definition of ϕ for convenience. The function ϕ is Prandtl's acceleration potential, as $\operatorname{grad}\phi$ gives the acceleration field of the flow. Taking the divergence of (3) and making use of (2), we obtain

$$\phi_{xx} + \phi_{yy} = 0. \tag{5}$$

Hence ϕ is a harmonic function of (x,y) for every t. Consequently a conjugate harmonic function $\psi(x,y,t)$ may be defined by the Cauchy–Riemann equations

$$\phi_x = \psi_y, \quad \phi_y = -\psi_x. \tag{6}$$

It then follows that in terms of the complex variable $z = x+iy$, the complex acceleration potential

$$f(z,t) = \phi(x,y,t) + i\psi(x,y,t) \tag{7}$$

is an analytic function of z at all times. Here, $i = \sqrt{-1}$ is the imaginary unit for the space variables (x,y). In terms of $f(z,t)$ and the complex velocity

$$w(z,t) = u(x,y,t) - iv(x,y,t), \tag{8}$$

(3) may also be written

$$\frac{\partial f}{\partial z} = \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial z}. \tag{9}$$

For given $f(z,t)$, w can be obtained from (9) by integration

$$w(z,t) = \frac{1}{U}\int_{-\infty}^{z} g\left(z_1, t+\frac{z_1-z}{U}\right) dz_1, \quad g(z,t) \equiv \frac{\partial f(z,t)}{\partial z}, \tag{10}$$

provided, of course, the integral exists. In the above, w is assumed to vanish at $z = -\infty$, the upstream infinity.

In unsteady flow, unlike steady flow, a vortex sheet will in general be shed from the solid plate to form a vortex wake, i.e. a surface across which the tangential velocity component is discontinuous, even in two-dimensional motions. The pressure p, however, is continuous everywhere inside the flow except across the solid plate. In particular, it is also continuous across the wake. Since ϕ is a linear function of p (see (4)), it follows that $f(z,t)$ must be a regular function of z for every t inside the flow except at the solid plate. Though the complex velocity w is also an analytic function of z, w may, however, admit discontinuities across the vortex wake, in addition to the singularities at the plate boundary. For this reason it is convenient to work with the acceleration potential; its application also results in considerable simplification in the analysis.

The boundary condition that the component of the flow velocity normal to the moving solid boundary must vanish may be linearized to give

$$v = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \quad \text{on} \quad y = \pm 0 \quad (-1 < x < 1). \tag{11}$$

The corresponding condition on $f(z,t)$ can be derived from (3) and (6) as

$$-\frac{\partial \psi}{\partial x} = \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 h(x,t) \quad \text{on} \quad y = \pm 0 \quad (-1 < x < 1). \tag{12}$$

It is seen from (11) and (12) that v and ψ are even in y, and hence from the Cauchy–Riemann equations, u and ϕ must be odd in y. Since ϕ is regular everywhere inside the flow, it follows that

$$\phi(x,0,t) = 0 \quad \text{for} \quad |x| > 1 \quad \text{and for all } t. \tag{13}$$

At the trailing edge of the plate, $z = 1$, the Kutta condition requires

$$|f(1,t)| < \infty, \quad \text{for all } t. \tag{14}$$

It should be pointed out, however, that in order to be possible to distinguish the trailing edge from the leading edge, the case of $U = 0$ should be treated as a limiting case of $U \to 0$ with the configuration $h(x,t)$ so given that the flow at $z = 1$ is in the direction of positive x. Furthermore, it is required that

$$f(z,t) \to 0 \quad \text{as} \quad |z| \to \infty; \quad w(z,t) \to 0 \quad \text{as} \quad z \to -\infty. \tag{15}$$

This completes the statement of the problem.

Of particular interest in the present work is the case in which the profile of the flexible plate has a progressive wave of arbitrary amplitude given by

$$h(x,t) = A(x)\cos(kx - \omega t + \epsilon) \quad \text{for} \quad |x| < 1, \tag{16}$$

where k is the wave-number, ω the circular frequency (which is taken to be positive throughout this work), ϵ an arbitrary phase angle and $A(x)$ the arbitrary amplitude of the wave motion. It is convenient to introduce the imaginary unit $j = \sqrt{-1}$ for the time variable t, e.g. $\exp(j\omega t) = \cos\omega t + j\sin\omega t$. The different notation j is used here so that it will not be confused with the imaginary unit i for the space variables x, y. (Thus, though $i^2 = -1$ and $j^2 = -1$, $ij \neq -1$.) In this notation the most general form of the simple harmonic motion of the flexible plate, including (16) as a special case, can be written

$$h(x,t) = h_1(x)\, e^{j\omega t} \quad \text{for} \quad |x| < 1, \tag{17}$$

where $h_1(x)$ is an arbitrary real function of x, but may in general be complex with respect to j. Taking the real part of the above expression for the eventual physical interpretation is of course understood.

By the conformal transformation

$$z = \tfrac{1}{2}(\zeta + \zeta^{-1}), \tag{18a}$$

the original z-plane, cut along the x-axis from $x = -1$ to $x = 1$, is mapped onto the region outside the unit circle $|\zeta| = 1$. On the unit circle, $\zeta = \exp(i\theta)$, and hence

$$x = \cos\theta. \tag{18b}$$

Since the plate has zero thickness, h and $\partial h/\partial x$ must be even functions of θ. We assume that $h_1(x)$ in (17) can be expanded in a Fourier cosine series

$$h(x,t) = \left[\tfrac{1}{2}\beta_0 + \sum_{n=1}^{\infty} \beta_n \cos n\theta\right] e^{j\omega t}, \tag{19a}$$

where
$$\beta_n = \frac{2}{\pi}\int_0^{\pi} h_1(x)\cos n\theta\, d\theta \quad (n = 0, 1, 2, \ldots). \tag{19b}$$

The coefficients β_n, like $h_1(x)$, may be complex in j. From (18b),

$$\partial/\partial x = -(\csc\theta)\,\partial/\partial\theta \quad \text{for} \quad y = 0, \quad |x| < 1.$$

Hence, from (19),

$$\frac{\partial h}{\partial x} = \frac{1}{\sin\theta}\sum_{n=1}^{\infty} n\beta_n \sin n\theta\, e^{j\omega t} = \left[\frac{\gamma_0}{2} + \sum_{n=1}^{\infty}\gamma_n \cos n\theta\right] e^{j\omega t}, \tag{20a}$$

from which it is readily seen that

$$\gamma_{n-1} - \gamma_{n+1} = 2n\beta_n \quad (n = 1, 2, \ldots). \tag{20b}$$

The coefficients γ_n can be solved from the above recursion formula to yield

$$\gamma_{2n} = 2\sum_{m=n}^{\infty}(2m+1)\beta_{2m+1}, \quad \gamma_{2n+1} = 2\sum_{m=n}^{\infty}(2m+2)\beta_{2m+2} \quad (n = 0, 1, 2, \ldots). \tag{20c}$$

Substituting (19), (20) in (11), we obtain

$$v(x, \pm 0, t) = -U\left[\tfrac{1}{2}\lambda_0 + \sum_{n=1}^{\infty}\lambda_n \cos n\theta\right] e^{j\omega t} \quad \text{on} \quad x = \cos\theta, \tag{21a}$$

where
$$\lambda_n = -(\gamma_n + j\sigma\beta_n), \quad \sigma = \omega/U, \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{21b}$$

The quantity $\sigma = \omega/U$ is called the reduced frequency referred to the half-chord which has been normalized to unity. If the chord is c, σ assumes the value $\omega c/2U$.

The solution for $f(z,t)$ satisfying conditions (21) and (12)–(15) has been obtained by Küssner & Schwarz (1940); the method may be briefly outlined as follows. If the acceleration potential $f(z,t)$ is invariant under the mapping (18), i.e. $f(z,t) = f\{z(\zeta),t\}$, then the solution must assume the form

$$f(z,t) = \phi + i\psi = iU^2\left[\frac{a_0}{\zeta+1} + \sum_{n=1}^{\infty}\frac{a_n}{\zeta^n}\right]e^{j\omega t} \quad \text{for} \quad |\zeta| > 1, \tag{22a}$$

where the coefficients a_n are real with respect to i so that condition (13) is satisfied. The term with a_0 in (22a) represents the singularity at the leading edge $\zeta = -1$; the infinite series represents an analytic function which, when $\partial h/\partial x$ has no discontinuity on the plate, is regular on and outside the unit circle $|\zeta| = 1$. To determine the a_n, we note that on the plate, $\zeta = e^{i\theta}$,

$$\phi = U^2\left[\frac{a_0}{2}\tan\frac{\theta}{2} + \sum_{n=1}^{\infty} a_n \sin n\theta\right]e^{j\omega t}, \tag{22b}$$

$$\psi = U^2\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta\right]e^{j\omega t}. \tag{22c}$$

Substituting (21) and (22c) in (12), we readily obtain

$$a_n = \lambda_n + \frac{j\sigma}{2n}(\lambda_{n-1} - \lambda_{n+1}) \quad \text{for} \quad n = 1,2,3,\ldots. \tag{23a}$$

The coefficient a_0 has to be determined by applying (10), the imaginary part of which becomes, in the present case of periodic motion,

$$v(x,y,t) = -\frac{1}{U}e^{-j\sigma x}\int_{-\infty}^{x} e^{j\sigma\xi}\frac{\partial\psi(\xi,y,t)}{\partial\xi}\,d\xi.$$

Choosing (x,y) to be any point on the plate (such as $x = -1+0$, $y = +0$), and applying conditions (21) and (22c), then by appropriate integration by parts and making use of (23a), we find that the infinite series cancel and the final result yields

$$a_0 = (\lambda_0 + \lambda_1)\,C(\sigma) - \lambda_1, \tag{23b}$$

$$C(\sigma) = \frac{K_1(j\sigma)}{K_0(j\sigma) + K_1(j\sigma)} = F(\sigma) + jG(\sigma), \tag{23c}$$

where K_0 and K_1 are the modified Bessel functions of the second kind. F and G are respectively the real and imaginary part of $C(\sigma)$, which is usually called Theodorsen's function (for its tabulated value, see Luke & Dengler 1951).

3. Hydrodynamic forces

The pressure difference across the plate, according to (4) and (22b), is

$$\begin{aligned}(\Delta p) &\equiv p(x,0-,t) - p(x,0+,t) = \rho[\phi(x,0+,t) - \phi(x,0-,t)]\\ &= \rho U^2\left[a_0\tan\tfrac{1}{2}\theta + 2\sum_{n=1}^{\infty} a_n\sin n\theta\right]e^{j\omega t},\end{aligned} \tag{24}$$

the positive sense of Δp being in the positive y-direction.

(1) *Lift*. The lift on the solid plate is

$$L = \int_{-1}^{1} (\Delta p)\, dx = \int_{0}^{\pi} (\Delta p) \sin\theta\, d\theta = \pi\rho U^2 (a_0 + a_1)\, e^{j\omega t}. \tag{25}$$

(2) *Moment*. The moment of force about the mid-chord, positive in the nose-up sense, is

$$M = -\int_{-1}^{1} (\Delta p)\, x\, dx = \tfrac{1}{2}\pi\rho U^2 (a_0 - a_2)\, e^{j\omega t}. \tag{26}$$

The real parts (with respect to j) of (25) and (26) give respectively the lift and moment on the plate due to the motion given by the real part of (19). These results are of course known in unsteady wing theory (e.g. see Robinson & Laurmann 1956).

(3) *Thrust*. The thrust, taken to be positive in the negative x-direction, acting on the solid plate is given by

$$T = \int_{-1}^{1} (\Delta p)_R \left(\frac{\partial h}{\partial x}\right)_R dx + T_S = T_P + T_S \tag{27}$$

in which the quantity with subscript R denotes the real part (with respect to j) of that quantity, and T_S represents the thrust due to the leading edge suction. Since the thrust contains the non-linear terms of small amplitude, in calculating the integral in (27) the real physical quantities, given by the real part of their complex representations, must be used directly. For the subsequent analysis it is convenient to separate the real and imaginary parts of the following functions. For $n = 0, 1, 2, \ldots,$

$$\left.\begin{aligned} \beta_n e^{j\omega t} &= B_n' + jB_n'', & \gamma_n e^{j\omega t} &= \Gamma_n' + j\Gamma_n'', \\ a_n e^{j\omega t} &= A_n' + jA_n'', & \lambda_n e^{j\omega t} &= \Lambda_n' + j\Lambda_n''. \end{aligned}\right\} \tag{28}$$

Then

$$(\Delta p)_R = \rho U^2 \left\{ A_0' \tan\tfrac{1}{2}\theta + 2 \sum_{n=1}^{\infty} A_n' \sin n\theta \right\}, \tag{29a}$$

$$\left(\frac{\partial h}{\partial x}\right)_R = \tfrac{1}{2}\Gamma_0' + \sum_{m=1}^{\infty} \Gamma_m' \cos m\theta. \tag{29b}$$

Substituting (29) into the integral for T_P in (27), we obtain

$$T_P = \rho U^2 \int_0^{\pi} \left\{ A_0'(1-\cos\theta) + \sum_{n=1}^{\infty} A_n' [\cos(n-1)\theta - \cos(n+1)\theta] \right\} \times \left\{ \tfrac{1}{2}\Gamma_0' + \sum_{m=1}^{\infty} \Gamma_m' \cos m\theta \right\} d\theta$$

$$= \tfrac{1}{2}\pi\rho U^2 \left\{ A_0'(\Gamma_0' - \Gamma_1') + \sum_{n=1}^{\infty} A_n' (\Gamma_{n-1}' - \Gamma_{n+1}') \right\}, \tag{30a}$$

or

$$T_P = \tfrac{1}{2}\pi\rho U^2 \left\{ A_0'(\Gamma_0' - \Gamma_1') + 2 \sum_{n=1}^{\infty} n A_n' B_n' \right\}. \tag{30b}$$

The last expression follows from (20b) and (28). T_P can still be expressed in another form by eliminating A'_n (for $n \geqslant 1$) from (30). Substituting (20b), (21b), (23) into (28), we have

$$\Gamma'_{n-1}-\Gamma'_{n+1} = 2nB'_n, \quad \Gamma''_{n-1}-\Gamma''_{n+1} = 2nB''_n, \quad \text{for} \quad n = 1,2,\ldots; \tag{31a}$$

$$A'_n = (\sigma^2/2n)(B'_{n-1}-B'_{n+1})+2\sigma B''_n-\Gamma'_n, \quad \text{for} \quad n = 1,2,\ldots; \tag{31b}$$

$$\begin{aligned} A'_0 &= (\Lambda'_0+\Lambda'_1)F(\sigma)-(\Lambda''_0+\Lambda''_1)G(\sigma)-\Lambda'_1 \\ &= (\sigma B''_0-\Gamma'_0)F(\sigma)-(\sigma B''_1-\Gamma'_1)[1-F(\sigma)] \\ &\qquad +[\sigma(B'_0+B'_1)+(\Gamma''_0+\Gamma''_1)]G(\sigma). \end{aligned} \tag{31c}$$

Hence $$\sum_{n=1}^{\infty} A'_n(\Gamma'_{n-1}-\Gamma'_{n+1}) = \sigma^2 \sum_{n=1}^{\infty} B'_n(B'_{n-1}-B'_{n+1})+4\sigma \sum_{n=1}^{\infty} nB'_nB''_n$$
$$-\sum_{n=1}^{\infty} \Gamma'_n(\Gamma'_{n-1}-\Gamma'_{n+1}) = \sigma^2 B'_0B'_1-\Gamma'_0\Gamma'_1+4\sigma\sum_{n=1}^{\infty} nB'_nB''_n.$$

Therefore

$$T_P = \tfrac{1}{2}\pi\rho U^2\left\{A'_0(\Gamma'_0-\Gamma'_1)-\Gamma'_0\Gamma'_1+\sigma^2B'_0B'_1+4\sigma\sum_{n=1}^{\infty} nB'_nB''_n\right\}. \tag{32}$$

This expression exhibits more explicitly the dependence of T_P on the reduced frequency σ since apart from A'_0 the coefficients B_n, Γ_n are seen to depend on σ only as a linear combination of $\sin\omega t$ and $\cos\omega t$.

The leading edge suction arises from the singular pressure at the leading edge, hence for its determination the non-linear terms in the expression for the pressure in the neighbourhood of the leading edge must be taken into account. The exact expression for the pressure is given by the Bernoulli equation

$$(p-p_\infty)/\rho = -\partial\Phi/\partial t-\tfrac{1}{2}(U+w_R)(U+\overline{w}_R)+\tfrac{1}{2}U^2,$$

where $\overline{w}_R$ denotes the complex conjugate of $w_R = u_R-iv_R$ with respect to i, the subscript R indicating the real part in j, and Φ is the velocity potential such that $u_R = \partial\Phi/\partial x$, $v_R = \partial\Phi/\partial y$. Now from (10) and (22),

$$w(z,t) = U\,e^{j\omega t-j\sigma z}\int_{-\infty}^{z} e^{j\sigma z_1}\frac{d}{dz_1}\left[\frac{ia_0}{\zeta_1+1}+\sum_{n=1}^{\infty}\frac{ia_n}{\zeta_1^n}\right]dz_1, \tag{33}$$

where $z_1 = \tfrac{1}{2}(\zeta_1+\zeta_1^{-1})$. Hence by integration by parts, it is readily seen that the asymptotic representation of $w(z,t)$ near the leading edge is

$$w(z,t) \cong \frac{iUa_0}{\zeta+1}e^{j\omega t}+O(1) \quad \text{as} \quad \zeta\to -1.$$

The part of w which is real in j thus reads

$$w_R(z,t) \cong \frac{iUA'_0}{\zeta+1}+O(1) \cong -\frac{UA'_0}{\{2(z+1)\}^{\frac{1}{2}}}+O(1) \quad \text{as} \quad z\to -1. \tag{34a}$$

From this result it can be seen that the velocity potential Φ and $\partial\Phi/\partial t$ remain bounded near the leading edge and hence

$$p-p_\infty \cong -\tfrac{1}{2}\rho w_R\overline{w}_R\left[1+O\left(\frac{1}{|w_R|}\right)\right] \quad \text{as} \quad z\to -1. \tag{34b}$$

Since $\partial\Phi/\partial t$ makes no contribution to p near the leading edge, the problem of the unsteady suction force hence reduces to the same problem for the steady motion in the sense that the time-dependent quantities appear only as parameters (or coefficients) in the expression for p. Let the x- and y-components of the singular force acting at the leading edge be denoted by X_S and Y_S, then by applying Blasius's formula to a small circle of radius ϵ surrounding the leading edge,

$$X_S - iY_S = \frac{i\rho}{2}\oint_\epsilon w_R^2 dz = -\tfrac{1}{2}\pi\rho U^2 A_0'^2.$$

Therefore $Y_S = 0$, and the leading edge thrust is

$$T_S = -X_S = \tfrac{1}{2}\pi\rho U^2 A_0'^2. \tag{35}$$

From (27), (32) and (35), the total thrust is

$$T = \tfrac{1}{2}\pi\rho U^2\left\{(A_0' + \Gamma_0')(A_0' - \Gamma_1') + \sigma^2 B_0' B_1' + 4\sigma \sum_{n=1}^{\infty} nB_n' B_n''\right\}. \tag{36}$$

Two limiting cases are of particular interest.

(i) $\sigma \to 0$. For $\sigma \ll 1$, $F(\sigma)$ and $G(\sigma)$ of (23c) can be expanded as

$$F(\sigma) = 1 - \tfrac{1}{2}\pi\sigma + O(\sigma^2 \log\sigma), \quad G(\sigma) = \sigma(\gamma + \log\tfrac{1}{2}\sigma) + O(\sigma^2\log\sigma), \tag{37}$$

where γ is Euler's constant, $\gamma = 0{\cdot}5771\ldots$. Substituting (31) and (37) in (36), we obtain

$$T = \tfrac{1}{2}\pi\rho U^2\{-(\sigma\log\tfrac{1}{2}\sigma)(\Gamma_0' + \Gamma_1')(\Gamma_0'' + \Gamma_1'') + O(\sigma)\} \quad \text{for} \quad \sigma \ll 1. \tag{38}$$

Since Γ_n' and Γ_n'' depend on σ only in the form of a linear combination of $\sin\omega t$ and $\cos\omega t$, T therefore tends to zero like $\sigma\log\sigma$ as $\sigma \to 0$. This is, of course, D'Alembert's paradox, as should be expected, that a rigid body immersed in a steady flow of an ideal fluid experiences no drag.

(ii) $\sigma \gg 1$. In this case, use of the asymptotic expansions of the Bessel functions in (23c) yields

$$F(\sigma) \sim \frac{1}{2}\left[1 + \frac{1}{8\sigma^2} + O(\sigma^{-4})\right], \quad G(\sigma) \sim -\frac{1}{8\sigma}\left[1 - \frac{11}{128\sigma^2} + O(\sigma^{-4})\right]. \tag{39}$$

Hence from (31c) and (36) we obtain

$$T \sim \tfrac{1}{2}\pi\rho U^2\{\sigma^2[B_0'B_1' + \tfrac{1}{4}(B_0'' - B_1'')^2] + O(\sigma)\} \quad \text{for} \quad \sigma \gg 1. \tag{40}$$

Thus, T behaves like σ^2 as $\sigma \to \infty$.

4. Power required; energy conservation

If the prescribed motion is to be maintained, then an external force equal and opposite to the pressure force across the plate must be applied. Over a segment of the plate of length dx, this external force is $-(\Delta p)_R dx$, acting in the positive y-direction. The power required to maintain the motion is equal to the time rate of work done by this external force, or

$$P = -\int_{-1}^{1} (\Delta p)_R \left(\frac{\partial h}{\partial t}\right)_R dx, \tag{41}$$

where $(\partial h/\partial t)_R$ denotes the real part of $\partial h/\partial t$ which, by (19) and (28), is

$$\left(\frac{\partial h}{\partial t}\right)_R = -\omega\left[\tfrac{1}{2}B_0'' + \sum_{m=1}^{\infty} B_m''\cos m\theta\right].$$

Substituting this equation and (29*a*) into (41) and integrating, we obtain

$$P = \tfrac{1}{2}\pi\rho U^2\omega\left\{A_0'(B_0''-B_1'') + \sum_{n=1}^{\infty} A_n'(B_{n-1}''-B_{n+1}'')\right\}. \tag{42a}$$

Upon elimination of A_n' from the above series, by using (31*b*), and after some rearrangement of the terms, we finally have

$$P = \tfrac{1}{2}\pi\rho U^3\sigma\left\{A_0'(B_0''-B_1'') + 2\sigma B_0''B_1'' - B_0''\Gamma_1' - B_1''\Gamma_0' + 2\sum_{n=1}^{\infty} nB_n'B_n'' \right.$$
$$\left. + \sigma^2\sum_{n=1}^{\infty}\frac{1}{2n}(B_{n-1}'-B_{n+1}')(B_{n-1}''-B_{n+1}'')\right\}. \tag{42b}$$

From this result it is noted that P vanishes with vanishing σ and behaves like σ^3 for large σ.

From the principle of conservation of energy, the power input P (or the energy put into the system per unit time) must be equal to the time rate of work done by the thrust, TU, plus the kinetic energy imparted to the fluid in unit time. Let the last quantity be denoted by E_W, then

$$P = TU + E_W. \tag{43}$$

Making use of (36) and (42*b*), we obtain

$$E_W = \tfrac{1}{2}\pi\rho U^3\left\{(\sigma B_0'' - \Gamma_0' - A_0')(\sigma B_1'' - \Gamma_1' + A_0') + \sigma^2(B_0''B_1'' - B_0'B_1') \right.$$
$$\left. - 2\sigma\sum_{n=1}^{\infty} nB_n'B_n'' + \sigma^3\sum_{n=1}^{\infty}\frac{1}{2n}(B_{n-1}'-B_{n+1}')(B_{n-1}''-B_{n+1}'')\right\}. \tag{44a}$$

From (21*b*) and (28),

$$\Lambda_n' = \sigma B_n'' - \Gamma_n', \quad \Lambda_n'' = -(\sigma B_n' + \Gamma_n'') \quad \text{for} \quad n = 0, 1, 2, \ldots, \tag{44b}$$

which can be used to rewrite (44*a*) as

$$E_W = \tfrac{1}{2}\pi\rho U^3\left\{(\Lambda_0' - A_0')(\Lambda_1' + A_0') - \sigma\sum_{n=1}^{\infty}\frac{1}{2n}(\Lambda_{n-1}'-\Lambda_{n+1}')(\Lambda_{n-1}''-\Lambda_{n+1}'')\right\}. \tag{44c}$$

The energy lost E_W can be determined by an alternative method. By the basic principle of hydrodynamics, E_W is equal to the time rate of work done by the pressure over the body surface, that is

$$E_W = -\int_{S_B} p\mathbf{n}.\mathbf{q}\,dS,$$

where $\mathbf{n}$ is the unit vector normal to the body surface S_B, pointing away from the fluid. At the leading edge where p and $\mathbf{q}$ are singular, S_B is replaced by a small circle ϵ around the leading edge. Thus

$$E_W = \oint_{\epsilon}(-p\mathbf{n}).\mathbf{q}\,dS_\epsilon - \int_{-1+0}^{1}(\Delta p)_R\, v_R(x,0,t)\,dx.$$

Using the solution ($34a, b$) in the first integral, we find that this integral is equal to $-T_S U$ where T_S is the leading edge suction (see (35)). The second integral is readily integrable; the final result is found to be identically ($44c$).

5. Average value of thrust and energy; efficiency of propulsion

Since we are primarily interested in the average values of the thrust and the power input over a single period $\tau = 2\pi/\omega$, we now define the time average of an arbitrary function of the time $g(t)$ by

$$\langle g(t)\rangle = \frac{1}{\tau}\int_0^\tau g(t)\,dt. \tag{45}$$

To determine the average values of the infinite series in (36), ($42b$) and (44), we write

$$\beta_n = \beta_n' + j\beta_n'' \quad (n = 0, 1, 2, \ldots), \tag{46a}$$

where β_n' and β_n'' are the real and imaginary parts of β_n. Then from (28),

$$B_n' = \beta_n' \cos\omega t - \beta_n'' \sin\omega t, \quad B_n'' = \beta_n'' \cos\omega t + \beta_n' \sin\omega t. \tag{46b}$$

Expressions similar to these may be written for Γ_n', Γ_n'', Λ_n', Λ_n''. By using the relations

$$\langle \cos^2\omega t\rangle = \langle \sin^2\omega t\rangle = \tfrac{1}{2}, \quad \langle \sin\omega t\,\cos\omega t\rangle = 0, \tag{47}$$

it is evident that

$$\langle B_n' B_n''\rangle = 0, \quad \langle (B_{n-1}' - B_{n+1}')(B_{n-1}'' - B_{n+1}'')\rangle = 0, \tag{48a}$$

$$\langle B_0' B_1'\rangle = \langle B_0'' B_1''\rangle = \tfrac{1}{2}(\beta_0'\beta_1' + \beta_0''\beta_1''). \tag{48b}$$

Therefore the time averages of the infinite series in (36), ($42b$) and (44) vanish and consequently

$$\langle T\rangle = \tfrac{1}{2}\pi\rho U^2 \langle (A_0' + \Gamma_0')(A_0' - \Gamma_1') + \sigma^2 B_0' B_1'\rangle, \tag{49}$$

$$\langle P\rangle = \tfrac{1}{2}\pi\rho U^3 \sigma \langle A_0'(B_0'' - B_1'') + 2\sigma B_0'' B_1'' - B_0''\Gamma_1' - B_1''\Gamma_0'\rangle, \tag{50}$$

$$\langle E_W\rangle = \tfrac{1}{2}\pi\rho U^3 \langle (\Lambda_0' - A_0')(\Lambda_1' + A_0')\rangle. \tag{51}$$

These expressions indicate that $\langle T\rangle$, $\langle P\rangle$, $\langle E_W\rangle$ depend on the time average of quantities which contain only five coefficients: Γ_0, Γ_1, B_0, B_1 and A_0. These equations are for the moment left in the above relatively simple form; their values for a specific case will be calculated in detail in § 7.

It is of interest to point out that $\langle E_W\rangle$ in general is positive definite. To show this result we first make use of ($31c$) and write (51) as

$$\langle E_W\rangle = \tfrac{1}{2}\pi\rho U^3 \langle (\Lambda_0' + \Lambda_1')^2 F(1-F) + (\Lambda_0' + \Lambda_1')(\Lambda_0'' + \Lambda_1'')(2F-1)G - (\Lambda_0'' + \Lambda_1'')^2 G^2\rangle. \tag{52}$$

We further write, as in (46),

$$\lambda_n = \lambda_n' + j\lambda_n'' \quad (n = 0, 1, 2, \ldots), \tag{53a}$$

and hence from (28)

$$\Lambda_n' = \lambda_n' \cos\omega t - \lambda_n'' \sin\omega t, \quad \Lambda_n'' = \lambda_n'' \cos\omega t + \lambda_n' \sin\omega t. \tag{53b}$$

Then from (47) it follows that

$$\langle(\Lambda_0'+\Lambda_1')^2\rangle = \langle(\Lambda_0''+\Lambda_1'')^2\rangle = \tfrac{1}{2}\{(\lambda_0'+\lambda_1')^2+(\lambda_0''+\lambda_1'')^2\},$$
$$\langle(\Lambda_0'+\Lambda_1')(\Lambda_0''+\Lambda_1'')\rangle = 0.$$

Therefore (52) becomes

$$\langle E_W\rangle = \tfrac{1}{4}\pi\rho U^3\{(\lambda_0'+\lambda_1')^2+(\lambda_0''+\lambda_1'')^2\}[F-(F^2+G^2)]. \tag{54}$$

From the known behaviour of the Theodorsen function (see (23c)), it can be shown that $F \geqslant (F^2+G^2)$ for $\sigma \geqslant 0$ and the equality holds only if $\sigma = 0$. Therefore it follows from (54) that in general $\langle E_W\rangle \geqslant 0$; it vanishes only in the special cases $\sigma = 0$ and $\lambda_0 = -\lambda_1$. The first case is the steady motion for which the result $E_W = 0$ is obvious. The second case includes the trivial limiting case of $\lambda_0 = \lambda_1 = 0$, but the general case with non-vanishing λ_0 and λ_1 is of more interest. As will be shown in the following section, the case of $\lambda_0 = -\lambda_1$, with $\lambda_0 \neq 0$, corresponds to the condition under which the circulation around the solid plate remains constant (zero) so that no trailing vortex wake will be shed from the plate, even in wavy motion.

From the above result it follows that $\langle P\rangle - \langle TU\rangle = \langle E_W\rangle$ is non-negative. Hence, when energy is taken out of the fluid ($\langle P\rangle$ negative), $\langle T\rangle$ cannot be positive. However, no definite statement, such as the above for $\langle E_W\rangle$, can be claimed separately for $\langle T\rangle$ and $\langle P\rangle$. When $\langle T\rangle \geqslant 0$, and hence $\langle P\rangle \geqslant U\langle T\rangle \geqslant 0$, we may define the average efficiency for producing useful thrust by

$$\eta = U\langle T\rangle/\langle P\rangle. \tag{55}$$

6. Circulation around the plate; vortex sheet strength

The vortex wake is assumed to lie along the x-axis downstream of the trailing edge. The strength of the vortex sheet at a point $P(x, 0)$ of the wake is denoted by $\gamma(x, t)$. The circulation $\delta\Gamma$ around a small rectangular circuit passing through the points $A(x, +0)$, $B(x+\delta x, +0)$, $C(x+\delta x, -0)$, $D(x, -0)$, in this order, is approximately

$$\delta\Gamma = \int_{ABCDA} \mathbf{q}\,.\,d\mathbf{s} = u(x, +0)\,\delta x - u(x, -0)\,\delta x,$$

which must be equal to the total vorticity within the circuit, $\gamma\delta x$. This statement may be regarded as the definition of γ. Since u is odd in y, we therefore have

$$\gamma(x,t) = 2u(x, +0, t). \tag{56}$$

Furthermore, at any point of the wake, $x > 1$,

$$\frac{\partial\gamma}{\partial t}+U\frac{\partial\gamma}{\partial x} = 2\left(\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}\right)_{y=+0} = 2\left(\frac{\partial\phi}{\partial x}\right)_{y=+0} = 0$$

by virtue of (3) and (13). Hence for $x > 1$, γ is a function of the single variable $(x-Ut)$ only. Thus, if the value of γ at the trailing edge is known, i.e.

$$\gamma(1,t) = 2u(1, +0, t) = g(t),$$

say, then

$$\gamma(x,t) = g\left(t-\frac{x-1}{U}\right) \quad \text{for} \quad x \geqslant 1. \tag{57}$$

In a co-ordinate system fixed in the fluid at infinity, the trailing edge of the plate travels a distance $-U\,dt$ along the x axis in the interval dt. If in the same time interval the circulation around the plate changes by the amount $d\Gamma$ from the value $\Gamma(t)$, then from the trailing edge there separates a vortex of strength $-d\Gamma$, which in turn is spread out into a vortex sheet of strength γ distributed over a length $U\,dt$ immediately behind the trailing edge. Hence

$$\frac{d\Gamma}{dt} = -U\gamma(1,t) = -2Uu(1,\,+0,\,t). \tag{58}$$

The circulation $\Gamma(t)$ around the plate can then be obtained by integration. Thus, once the value of u at the trailing edge is known, the strength of the vortex wake and the circulation around the plate can be determined from (57) and (58).

To calculate u, we first substitute ($22a$) in (10) and integrate by parts, giving

$$\begin{aligned} w(z,t) &= \frac{1}{U}e^{-j\sigma z}\int_{-\infty}^{z} e^{j\sigma z_1}\frac{\partial f(z_1,t)}{\partial z_1}\,dz_1 \\ &= \frac{1}{U}\left\{f(z,t) - j\sigma\, e^{-j\sigma z}\int_{-\infty}^{z} e^{j\sigma z_1} f(z_1,t)\,dz_1\right\}. \end{aligned}$$

Now $\phi = Re\,f = 0$ on $y = 0$, $|x| > 1$; hence by separating the real and imaginary parts (in $i = \sqrt{-1}$), making use of (22), and by taking $\theta = \cos^{-1}x$ as the integration variable over the plate, we obtain

$$\begin{aligned} u(1,\,+0,\,t) &= -\frac{j\sigma}{U}e^{-j\sigma}\int_{-1}^{1} e^{j\sigma x}\,\phi(x,\,+0,\,t)\,dx \\ &= -j\sigma U\,e^{j(\omega t-\sigma)}\int_0^{\pi} e^{j\sigma\cos\theta}\left[\frac{a_0}{2}(1-\cos\theta) + \sum_{n=1}^{\infty} a_n \sin n\theta\,\sin\theta\right]d\theta. \end{aligned}$$

Substituting from ($23a$) into the above integral and integrating the terms with the factor $(j\sigma/2n)$ by parts, we find

$$\begin{aligned} &u(1,\,+0,\,t)\,e^{-j(\omega t-\sigma)}/(\tfrac{1}{2}j\sigma U) \\ &\quad= -\int_0^{\pi} e^{j\sigma\cos\theta}\left\{a_0(1-\cos\theta) + 2\sum_{n=1}^{\infty}\left[\lambda_n + \frac{j\sigma}{2n}(\lambda_{n-1}-\lambda_{n+1})\right]\sin n\theta\,\sin\theta\right\}d\theta \\ &\quad= -\int_0^{\pi} e^{j\sigma\cos\theta}\left\{a_0(1-\cos\theta) + \sum_{n=1}^{\infty}\lambda_n[\cos(n-1)\theta - \cos(n+1)\theta]\right. \\ &\qquad\qquad\left. + \sum_{n=1}^{\infty}(\lambda_{n-1}-\lambda_{n+1})\cos n\theta\right\}d\theta. \end{aligned}$$

The infinite series in the above integrand terminates after rearrangement. Now (see Watson 1944)

$$\int_0^{\pi} e^{j\sigma\cos\theta}\cos n\theta\,d\theta = \pi j^n J_n(\sigma) = \pi I_n(j\sigma),$$

where J_n is the Bessel function and I_n the modified Bessel function, both of the first kind. Hence

$$u(1,\,+0,\,t) = -\tfrac{1}{2}\pi j\sigma U\,e^{j(\omega t-\sigma)}\{(\lambda_1+a_0)\,I_0(j\sigma) + (\lambda_0-a_0)\,I_1(j\sigma)\}.$$

By using (23) and the relation (see Watson 1944, p. 80)

$$I_0(z)\,K_1(z)+I_1(z)\,K_0(z)=\frac{1}{z},$$

the above expression may be written

$$u(1,\,+0,\,t)=-\tfrac{1}{2}\pi U\,e^{j(\omega t-\sigma)}\frac{\lambda_0+\lambda_1}{K_0(j\sigma)+K_1(j\sigma)}. \tag{59}$$

If $\lambda_0+\lambda_1=0$, then it follows from (57)–(59) that the circulation around the plate remains constant for all time and hence the vortex wake disappears. Furthermore, it can be shown that $K_0(j\sigma)+K_1(j\sigma)$ has no zero for real σ (see Erdélyi & Kermack 1945). Therefore the necessary and sufficient condition for constant circulation around the flexible plate is

$$\lambda_0+\lambda_1=0, \tag{60a}$$

which may be expressed, by using (20) and (21), in terms of the β_n as

$$j\sigma(\beta_0+\beta_1)+2\sum_{n=1}^{\infty}n\beta_n=0. \tag{60b}$$

The motion for which the β_n's (see (19)) satisfy (60*b*) is the one in which no trailing vortices will be shed.

The circulation around the plate, from (58) and (59), is

$$\Gamma(t)=\pi U\,e^{j(\omega t-\sigma)}\frac{\lambda_0+\lambda_1}{j\sigma[K_0(j\sigma)+K_1(j\sigma)]}. \tag{61}$$

In the limit of steady motion, $\omega\to 0$, $\sigma\to 0$, the denominator of the above expression tends to unity, and $\lambda_n\to-\gamma_n$ (see (21*b*)), hence $\Gamma\to-\pi U(\gamma_0+\gamma_1)$. Also, as $\sigma\to 0$, $a_n\to\lambda_n$ for $n=0,1,2,\ldots$ (see (23)), and so from (25) the lift becomes

$$L=\pi\rho U^2(\lambda_0+\lambda_1)=-\pi\rho U^2(\gamma_0+\gamma_1)=\rho U\Gamma,$$

in agreement with the theorem of Joukowski.

7. The flexible plate with progressive waves

Consider in particular the motion of a flexible plate which consists of a train of progressive waves of wavelength $\lambda=2\pi/k$ and frequency $\omega/2\pi$, as given by

$$h(x,t)=\left[\sum_{m=0}^{\infty}b_m\exp(j\epsilon_m)\,x^m\right]e^{j(\omega t-kx)}\quad(-1<x<1), \tag{62a}$$

where b_m are taken to be real coefficients, ϵ_m are constant phase angles, the amplitude of the wave being taken as an arbitrary function of x as represented by the series. Without losing generality we may take $\epsilon_0=0$. The above equation may also be written

$$h(x,t)=\sum_{m=0}^{\infty}b_m\,j^m\left(\frac{\partial}{\partial k}\right)^m\exp\{j(\omega t-kx+\epsilon_m)\}\quad(-1<x<1). \tag{62b}$$

In terms of the variable θ defined by (18*b*), we may write (Watson 1944, p. 22)

$$e^{-jkx}=e^{-jk\cos\theta}=J_0(k)+2\sum_{n=1}^{\infty}(-j)^nJ_n(k)\cos n\theta, \tag{63}$$

where $J_n(k)$ is the Bessel function of the first kind. Thus

$$h(x,t) = \sum_{m=0}^{\infty} b_m j^m \exp(j\epsilon_m) \left[J_0^{(m)}(k) + 2 \sum_{n=1}^{\infty} (-j)^n J_n^{(m)}(k) \cos n\theta \right] e^{j\omega t}, \quad (64)$$

where $J_n^{(m)}(k)$ stands for $d^m J_n(k)/dk^m$. If the wave propagates in the negative x-direction, k assumes negative values; this can be obtained by the continuation

$$J_n(k) = (-)^n J_n(-k), \quad J_n^{(m)}(k) = (-)^{m+n} J_n^{(m)}(-k). \quad (65)$$

Comparing (64) with (19), we find that

$$\beta_n = 2 \sum_{m=0}^{\infty} b_m j^{m-n} \exp(j\epsilon_m) J_n^{(m)}(k) \quad (n = 0, 1, 2, \ldots). \quad (66)$$

Substituting (66) in (20c) and using the relations (Erdélyi 1953, vol. 2, p. 99)

$$\left. \begin{aligned} kJ_{2n}(k) &= 2 \sum_{\nu=n}^{\infty} (-)^{\nu-n} (2\nu+1) J_{2\nu+1}(k) \quad (n = 0, 1, 2, \ldots), \\ kJ_{2n+1}(k) &= 2 \sum_{\nu=n}^{\infty} (-)^{\nu-n} (2\nu+2) J_{2\nu+2}(k) \quad (n = 0, 1, 2, \ldots), \end{aligned} \right\} \quad (67)$$

we obtain, for $n = 0, 1, 2, \ldots$,

$$\gamma_n = 2j^{-(n+1)} \sum_{m=0}^{\infty} b_m j^m \exp(j\epsilon_m) \left(\frac{d}{dk}\right)^m [kJ_n(k)]. \quad (68)$$

Substituting (66) in (28), we find, for $n = 0, 1, 2, \ldots$,

$$\left. \begin{matrix} B'_{2n} \\ -B''_{2n+1} \end{matrix} \right\} = 2(-)^n \sum_{m=0}^{\infty} D_{\mathrm{I}}(m,t) \begin{Bmatrix} J_{2n}(k) \\ J_{2n+1}(k) \end{Bmatrix}, \quad (69a)$$

$$\left. \begin{matrix} B''_{2n} \\ B'_{2n+1} \end{matrix} \right\} = 2(-)^n \sum_{m=0}^{\infty} D_{\mathrm{II}}(m,t) \begin{Bmatrix} J_{2n}(k) \\ J_{2n+1}(k) \end{Bmatrix}, \quad (69b)$$

where D_{I} and D_{II} denote the differential operators

$$\begin{aligned} D_{\mathrm{I}}(m,t) &= (-)^m \left[b_{2m} \cos(\omega t + \epsilon_{2m}) \left(\frac{\partial}{\partial k}\right)^{2m} - b_{2m+1} \sin(\omega t + \epsilon_{2m+1}) \left(\frac{\partial}{\partial k}\right)^{2m+1} \right], \\ D_{\mathrm{II}}(m,t) &= (-)^m \left[b_{2m} \sin(\omega t + \epsilon_{2m}) \left(\frac{\partial}{\partial k}\right)^{2m} + b_{2m+1} \cos(\omega t + \epsilon_{2m+1}) \left(\frac{\partial}{\partial k}\right)^{2m+1} \right]. \end{aligned} \quad (70)$$

Similarly, from (68) and (28) we find, for $n = 0, 1, 2, \ldots$,

$$\left. \begin{matrix} \Gamma'_{2n} \\ -\Gamma''_{2n+1} \end{matrix} \right\} = 2(-)^n \sum_{m=0}^{\infty} D_{\mathrm{II}}(m,t) \begin{Bmatrix} kJ_{2n}(k) \\ kJ_{2n+1}(k) \end{Bmatrix}, \quad (71a)$$

$$\left. \begin{matrix} -\Gamma''_{2n} \\ -\Gamma'_{2n+1} \end{matrix} \right\} = 2(-)^n \sum_{m=0}^{\infty} D_{\mathrm{I}}(m,t) \begin{Bmatrix} kJ_{2n}(k) \\ kJ_{2n+1}(k) \end{Bmatrix}, \quad (71b)$$

Upon substitution of the above relations in (31c), we obtain

$$A'_0 = 2 \sum_{m=0}^{\infty} \{ D_{\mathrm{II}}(m,t)\, \Omega(\sigma,k) + D_{\mathrm{I}}(m,t)\, Q(\sigma,k) \}, \quad (72a)$$

where

$$\Omega(\sigma,k) = (\sigma - k)\{F(\sigma) J_0(k) + G(\sigma) J_1(k)\}, \quad (72b)$$

$$Q(\sigma,k) = (\sigma - k)\{[1 - F(\sigma)] J_1(k) + G(\sigma) J_0(k)\}. \quad (72c)$$

The other coefficients (namely $A'_1, A'_2, \ldots, A''_n, \ldots$) do not appear in $\langle T \rangle$ and $\langle P \rangle$; they will not be given here.

Finally, when the above expressions for B_0, B_1, Γ_0, Γ_1, A_0 are substituted in (49), we have

$$\frac{\langle T\rangle}{\pi\rho U^2} = 2\left\langle \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}[D_{\mathrm{II}}(m,t)\,\Omega_* + D_{\mathrm{I}}(m,t)\,Q]\,[D_{\mathrm{II}}(r,t)\,\Omega + D_{\mathrm{I}}(r,t)\,Q_*]\right\rangle + 2\sigma^2\left\langle \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}[D_{\mathrm{I}}(m,t)\,J_0(k)]\,[D_{\mathrm{II}}(r,t)\,J_1(k)]\right\rangle, \tag{73a}$$

where
$$\Omega_* = \Omega(\sigma,k) + kJ_0(k), \quad Q_* = Q(\sigma,k) + kJ_1(k). \tag{73b}$$

By making use of the results

$$\langle\sin(\omega t+\epsilon)\sin(\omega t+\delta)\rangle = \langle\cos(\omega t+\epsilon)\cos(\omega t+\delta)\rangle = \tfrac{1}{2}\cos(\epsilon-\delta),$$
$$\langle\sin(\omega t+\epsilon)\cos(\omega t+\delta)\rangle = \tfrac{1}{2}\sin(\epsilon-\delta),$$

we obtain, after some straightforward manipulation,

$$\begin{aligned}\frac{\langle T\rangle}{\pi\rho U^2} = {} & \left(\sum_{n=0,2,4,\ldots}^{\infty}(-)^{\frac{1}{2}n}\sum_{r=0,2,\ldots}^{n} + \sum_{n=2,4,\ldots}^{\infty}(-)^{\frac{1}{2}n-1}\sum_{r=1,3,\ldots}^{n-1}\right)b_r b_{n-r} \\ & \times\{[\Omega^{(r)}\Omega_*^{(n-r)} + Q^{(r)}Q_*^{(n-r)}]\cos(\epsilon_r-\epsilon_{n-r}) \\ & + [\Omega^{(r)}Q^{(n-r)} + \Omega_*^{(r)}Q_*^{(n-r)} + \sigma^2 J_1^{(r)}J_0^{(n-r)}]\sin(\epsilon_r-\epsilon_{n-r})\} \\ & + \sum_{n=1,3,\ldots}^{\infty}(-)^{\frac{1}{2}n-\frac{1}{2}}\sum_{r=0,2,\ldots}^{n-1} b_r b_{n-r}\{[\Omega^{(r)}\Omega_*^{(n-r)} + \Omega_*^{(r)}\Omega^{(n-r)} + Q^{(r)}Q_*^{(n-r)} \\ & + Q_*^{(r)}Q^{(n-r)}]\sin(\epsilon_r-\epsilon_{n-r}) + [Q^{(r)}\Omega^{(n-r)} + Q_*^{(r)}\Omega_*^{(n-r)} - \Omega^{(r)}Q^{(n-r)} \\ & - \Omega_*^{(r)}Q_*^{(n-r)} + \sigma^2(J_0^{(r)}J_1^{(n-r)} - J_1^{(r)}J_0^{(n-r)})]\cos(\epsilon_r-\epsilon_{n-r})\},\end{aligned} \tag{74}$$

where the superscript numerals in the parentheses denote the order of differentiation with respect to k, e.g. $\Omega^{(n)} = \partial^n\Omega/\partial k^n$.

To calculate $\langle P\rangle$ it is convenient to use the relation $\langle P\rangle = \langle TU\rangle + \langle E_W\rangle$, since $\langle E_W\rangle$ can be readily derived from (54). From the definitions (21b) and (53a), we obtain by using (66) and (68) the following

$$\left.\begin{matrix}\lambda_0' \\ -\lambda_1''\end{matrix}\right\} = 2\sum_{m=0}^{\infty} D_{\mathrm{II}}(m,0)\begin{Bmatrix}(\sigma-k)\,J_0(k) \\ (\sigma-k)\,J_1(k)\end{Bmatrix},$$

$$\left.\begin{matrix}\lambda_0'' \\ \lambda_1'\end{matrix}\right\} = -2\sum_{m=0}^{\infty} D_{\mathrm{I}}(m,0)\begin{Bmatrix}(\sigma-k)\,J_0(k) \\ (\sigma-k)\,J_1(k)\end{Bmatrix}.$$

Substituting these expressions into (54) and rearranging the terms, we find that

$$\begin{aligned}\frac{\langle E_W\rangle}{\pi\rho U^3} = {} & [F-(F^2+G^2)]\left\{\left(\sum_{n=0,2,\ldots}^{\infty}(-)^{\frac{1}{2}n}\sum_{r=0,2,\ldots}^{n} + \sum_{n=2,4,\ldots}^{\infty}(-)^{\frac{1}{2}n-1}\sum_{r=1,3,\ldots}^{n-1}\right)b_r b_{n-r}\right. \\ & \times[(N_0^{(r)}N_0^{(n-r)} + N_1^{(r)}N_1^{(n-r)})\cos(\epsilon_r-\epsilon_{n-r}) - 2N_0^{(r)}N_1^{(n-r)}\sin(\epsilon_r-\epsilon_{n-r})] \\ & + 2\sum_{n=1,3,\ldots}^{\infty}(-)^{\frac{1}{2}n-\frac{1}{2}}\sum_{r=0,2,\ldots}^{n-1} b_r b_{n-r}[(N_0^{(r)}N_0^{(n-r)} + N_1^{(r)}N_1^{(n-r)})\sin(\epsilon_r-\epsilon_{n-r}) \\ & \left. + (N_0^{(r)}N_1^{(n-r)} - N_1^{(r)}N_0^{(n-r)})\cos(\epsilon_r-\epsilon_{n-r})]\right\},\end{aligned} \tag{75a}$$

where
$$N_0(\sigma,k) = (\sigma-k)\,J_0(k), \quad N(\sigma,k) = (\sigma-k)\,J_1(k). \tag{75b}$$

With $\langle T \rangle$ and $\langle E_W \rangle$ determined as above, the average power required is then

$$\langle P \rangle = U\langle T \rangle + \langle E_W \rangle. \tag{76}$$

If the chord of the plate is c, then the above results hold valid if $\sigma = \omega c/2U$, $k = \pi c/\lambda$ (λ being the wavelength), $\langle T \rangle/\pi\rho U^2$ is replaced by $\langle T \rangle/\pi\rho U^2(\frac{1}{2}c)$, and $\langle E_W \rangle/\pi\rho U^3$ by $\langle E_W \rangle/\pi\rho U^3(\frac{1}{2}c)$.

If the series in (62a), which represents the amplitude function of the plate motion, has but a finite number of terms, then both series in (74) and (75) will terminate. In particular, we shall consider the following special example.

8. Waving plate with linearly varying amplitude

In order to exhibit the effects of the non-uniform amplitude of the progressing wave along the plate and the effect of the different phase angles, we consider the relatively simple case

$$\begin{aligned} h(x,t) &= b_0 \cos(\omega t - kx) + b_1 \cos(\omega t - kx + \epsilon) \\ &= Re\{[b_0 + b_1 x\, e^{j\epsilon}]\, e^{j(\omega t - kx)}\}, \quad \text{for} \quad -1 < x < 1. \end{aligned} \tag{77}$$

This is a special case of the general motion (62) with $b_2 = b_3 = \ldots = 0$, $\epsilon_0 = 0$, $\epsilon_1 = \epsilon$. From (74) it is readily verified that $\langle T \rangle$ now reduces to

$$\frac{\langle T \rangle}{\pi\rho U^2} = b_0^2 T_1(\sigma,k) + b_1^2 T_2(\sigma,k) + b_0 b_1[T_3(\sigma,k)\cos\epsilon + T_4(\sigma,k)\sin\epsilon], \tag{78}$$

where
$$T_1(\sigma,k) = \Omega(\sigma,k)\,\Omega_*(\sigma,k) + Q(\sigma,k)\,Q_*(\sigma,k),$$

$$T_2(\sigma,k) = \frac{\partial\Omega}{\partial k}\frac{\partial\Omega_*}{\partial k} + \frac{\partial Q}{\partial k}\frac{\partial Q_*}{\partial k},$$

$$T_3(\sigma,k) = Q\frac{\partial\Omega}{\partial k} + Q_*\frac{\partial\Omega_*}{\partial k} - \Omega\frac{\partial Q}{\partial k} - \Omega_*\frac{\partial Q_*}{\partial k} + \sigma^2\left[J_0^2 + J_1^2 - \frac{1}{k}J_0 J_1\right],$$

$$T_4(\sigma,k) = -\frac{\partial}{\partial k}T_1(\sigma,k).$$

The functions Ω, Q are given by (72) and Ω_*, Q_* by (73).

From (75) we derive, for the present case,

$$\frac{\langle E_W \rangle}{\pi\rho U^3} = b_0^2 W_1(\sigma,k) + b_1^2 W_2(\sigma,k) + b_0 b_1[W_3(\sigma,k)\cos\epsilon + W_4(\sigma,k)\sin\epsilon], \tag{79}$$

where
$$W_1(\sigma,k) = [F - (F^2+G^2)]\,[N_0^2 + N_1^2],$$

$$W_2(\sigma,k) = [F - (F^2+G^2)]\left[\left(\frac{\partial N_0}{\partial k}\right)^2 + \left(\frac{\partial N_1}{\partial k}\right)^2\right],$$

$$W_3(\sigma,k) = 2[F - (F^2+G^2)]\left[N_0\frac{\partial N_1}{\partial k} - N_1\frac{\partial N_0}{\partial k}\right],$$

$$W_4(\sigma,k) = -\frac{\partial}{\partial k}W_1(\sigma,k).$$

The functions N_0 and N_1 are given by (75b).

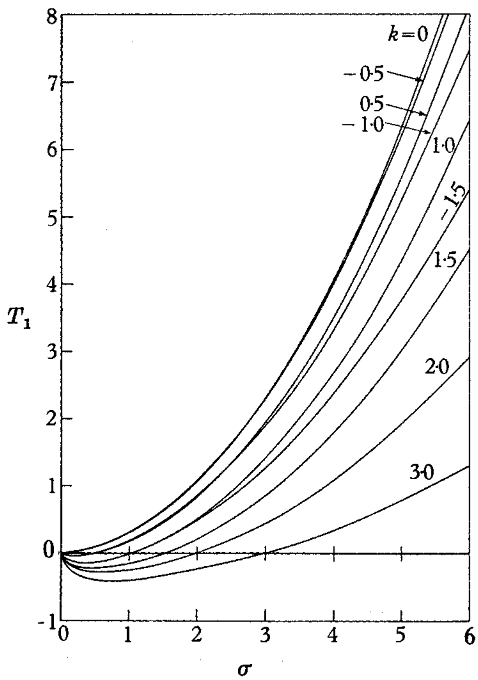

FIGURE 1

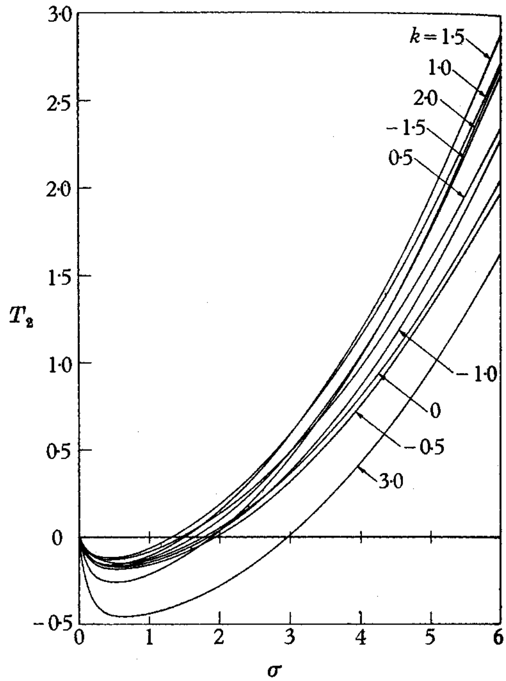

FIGURE 2

FIGURE 1. Variation of the thrust-coefficient function $T_1(\sigma, k)$ with the reduced frequency σ at several circular wave-numbers k (cf. statement below equation (76)).

FIGURE 2. Variation of the thrust-coefficient function $T_2(\sigma, k)$ with the reduced frequency σ at several values of k.

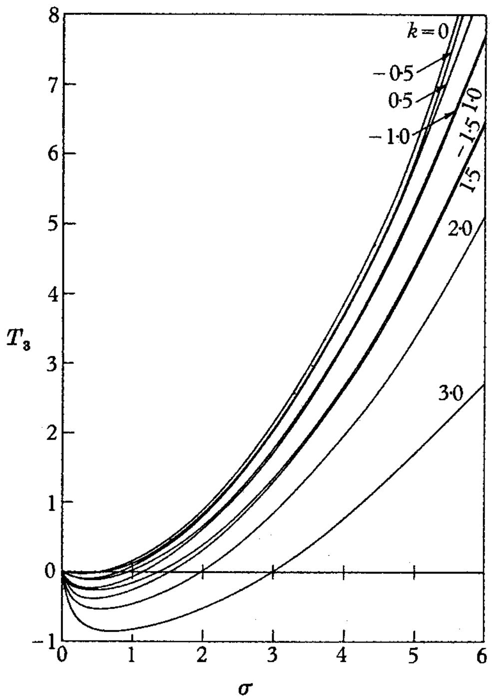

FIGURE 3

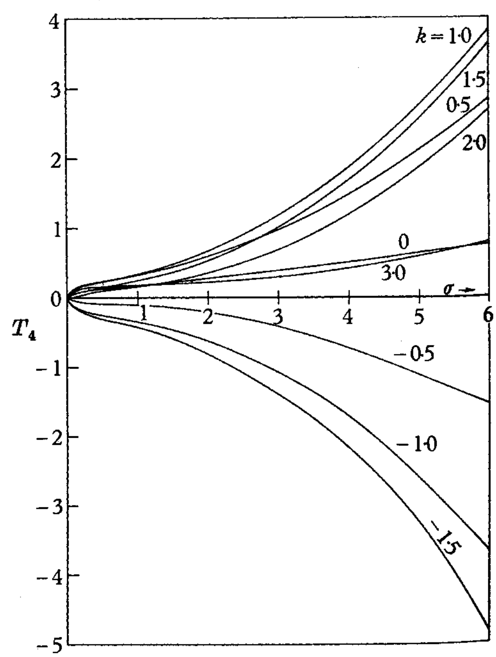

FIGURE 4

FIGURE 3. Variation of the thrust-coefficient function $T_3(\sigma, k)$ with the reduced frequency σ at several values of k.

FIGURE 4. Variation of the thrust-coefficient function $T_4(\sigma, k)$ with the reduced frequency σ at several values of k.

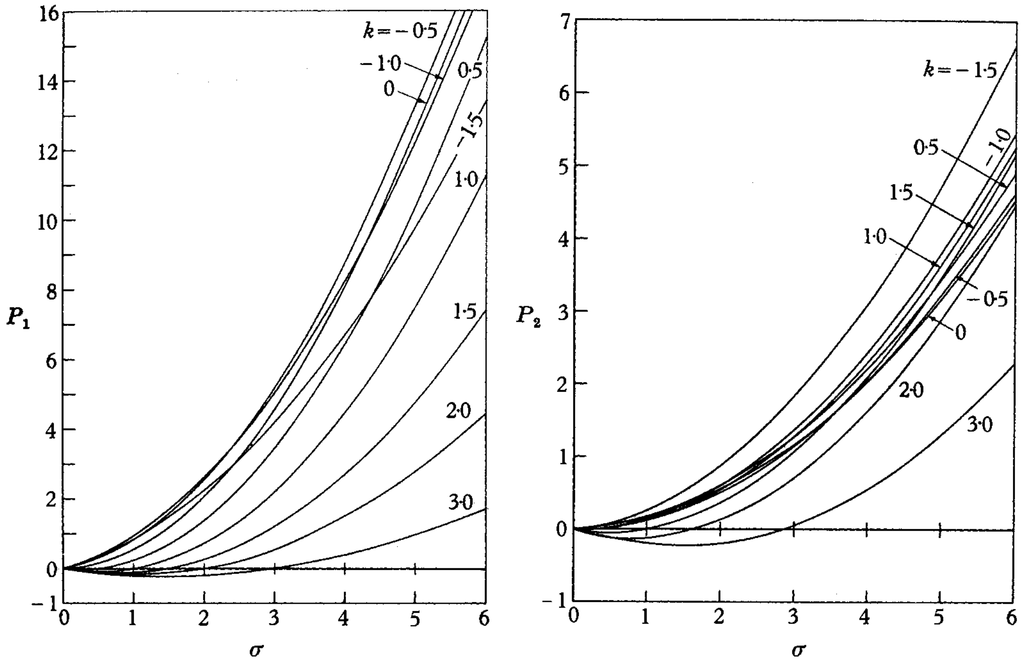

FIGURE 5. Variation of the power-coefficient functions $P_1(\sigma, k)$ and $P_2(\sigma, k)$ with the reduced frequency σ at several values of k.

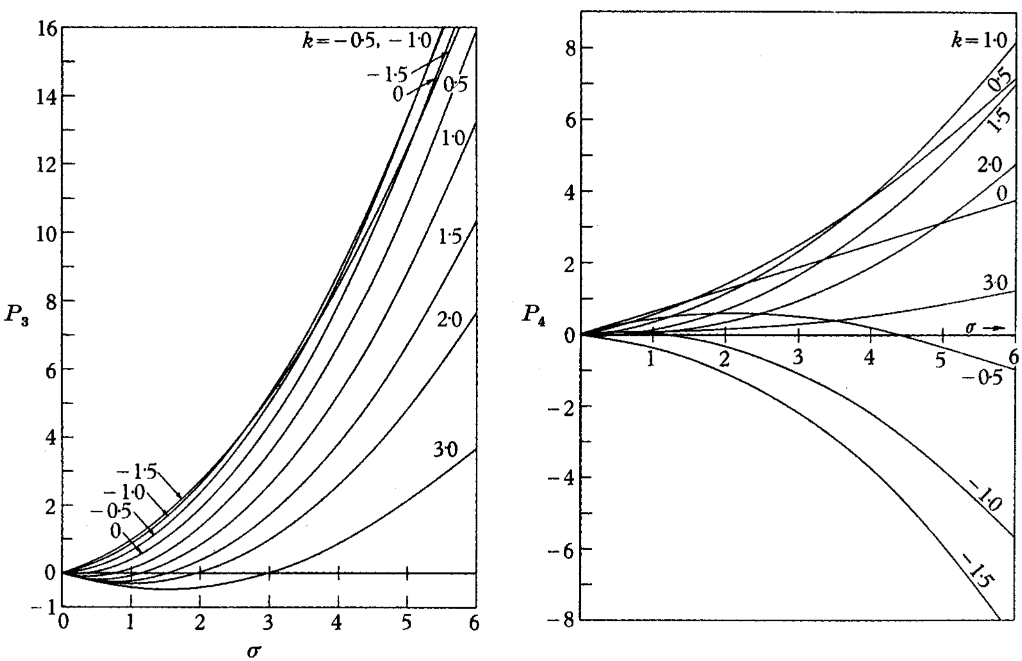

FIGURE 6. Variation of the power-coefficient functions $P_3(\sigma, k)$ and $P_4(\sigma, k)$ with the reduced frequency σ at several values of k.

The average power input, from (76), is therefore

$$\frac{\langle P \rangle}{\pi \rho U^3} = b_0^2 P_1(\sigma, k) + b_1^2 P_2(\sigma, k) + b_0 b_1 [P_3(\sigma, k) \cos \epsilon + P_4(\sigma, k) \sin \epsilon], \tag{80}$$

where $$P_i(\sigma, k) = T_i(\sigma, k) + W_i(\sigma, k) \quad (i = 1, 2, 3, 4).$$

In figures 1–6 the above results for T_i and P_i are plotted versus the reduced frequency σ in the range $0 \leqslant \sigma \leqslant 6$ for the wave-numbers $k = -1{\cdot}5, -1, -0{\cdot}5, 0, 0{\cdot}5, 1, 1{\cdot}5, 2, 3$. The results show that T_i and P_i vanish at $\sigma = 0$, and, for large values of σ, the magnitudes of T_i and P_i behave like σ^2 and σ^3, respectively. For any given values of b_0 and b_1, the average thrust and the power required can be readily obtained by simple linear combination of these functions. Particularly simple are the cases (1) $b_0 \neq 0, b_1 = 0$, and (2) $b_0 = 0, b_1 \neq 0$; the solution for these cases, apart from a proportionality factor, are simply T_1, P_1 and T_2, P_2.

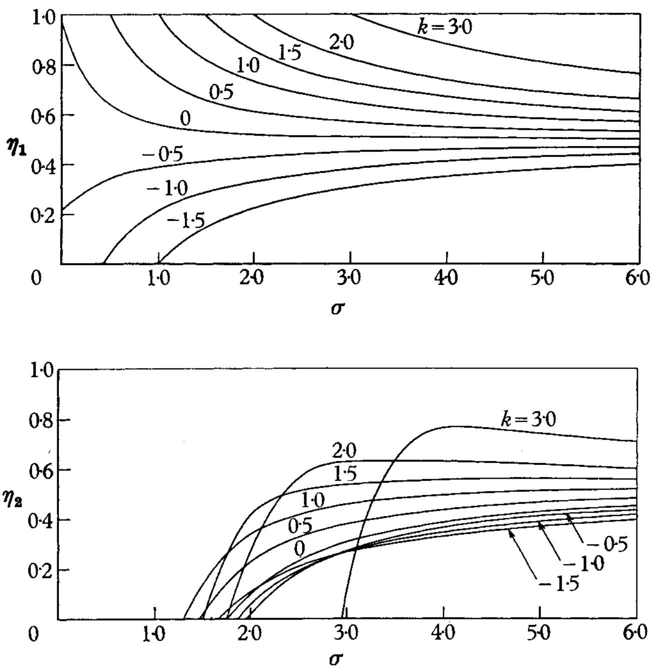

FIGURE 7. Swimming efficiency for two configurations, η_1 for a uniform wave amplitude, and η_2 for a linearly varying amplitude symmetrical about the plate centre.

The first case ($b_0 \neq 0$, $b_1 = 0$) represents a waving plate with a uniform wave amplitude. For a given positive k, implying that the wave travels toward the tail, both the thrust and the power required vanish at $\sigma = k$, when the wave velocity is equal to the free-stream velocity since the wave velocity is then

$$c = \omega / k = U\sigma / k = U.$$

This means that this wave form becomes frozen with respect to the fluid at infinity so that the plate merely travels along a sinusoidal path fixed in the space. As can be verified, this motion creates no circulation around the waving plate.

For positive k, the thrust $\langle T\rangle$ and the power required $\langle P\rangle$ are negative for $0 < \sigma < k$ and positive for $\sigma > k$. Negative values of $\langle P\rangle$ mean that energy is taken out of the fluid; this is possible only if $k > 0$. For a rigid plate in transverse oscillation ($k = 0$), $\langle P\rangle$ is non-negative, in agreement with the remark by Kármán that energy can be taken out the fluid only in the case of an oscillatory motion with at least two degrees of freedom. For a flexible plate with $k \neq 0$ the degrees of freedom are infinite in number. At every fixed σ, both T_1 and P_1 decrease with increasing positive k. In the range of negative k, corresponding to wave propagation toward the head, it is noted that T_1 and P_1 have the general trend of decreasing with decreasing k. Thus, T_1 and P_1 reach their maximum values near $k = 0$ for each given σ. Perhaps of more significance is the efficiency which becomes in this case $\eta_1 = T_1/P_1$. The values of η_1, plotted in figure 7, indicate that in the range of positive thrust ($T_1 > 0$) η_1 increases with increasing k (negative as well as positive) for fixed σ. This implies that the percentage rate of decrease in T_1 is slower than that in P_1 with respect to increasing k, for all k. Consequently, from the viewpoint of the efficiency, it is advantageous to have the waves propagating towards the rear. This result is interpreted here as affording a qualitative explanation of the observation that fishes in nature usually have the wave form of their body propagating from the head to the tail.

The functions T_2 and P_2 behave in much the same manner as T_1 and P_1, as may be seen from the figures. The function $\eta_2 = T_2/P_2$ is plotted in figure 7.

In the case of $U \to 0$, with $\omega = \sigma U$ fixed, the limiting solution becomes

$$\frac{\langle T\rangle}{\pi\rho\omega^2} = \frac{b_0^2}{4}(J_0^2+J_1^2)+\frac{b_1^2}{4}\left[\left(J_0-\frac{J_1}{k}\right)^2+J_1^2\right] + \frac{b_0 b_1}{2}\left[\left(J_0^2+J_1^2-\frac{1}{k}J_0 J_1\right)\cos\epsilon+\frac{1}{k}J_1^2\sin\epsilon\right],$$

$$\langle P\rangle \to 2\langle E_W\rangle \to 2U\langle T\rangle.$$

9. The basic mechanism of swimming

From the basic principle of action and reaction in mechanics, it is to be expected that, when the plate attains a forward momentum as it swims through a fluid, the fluid must be pushed in the backward direction with a net total momentum equal and opposite to that of the action. From the resulting flow picture it can be seen that the momentum of reaction is concentrated in the vortex wake and appears in the form of a jet of fluid expelled from the plate. This mechanism has been elucidated, for the simple case of a rigid plate in transverse oscillation, by Kármán & Burgers (1943, p. 308). It is not difficult to illustrate this mechanism for the present more general case by the following consideration.

Consider again the simple case $b_0 \neq 0$, $b_1 = b_2 = \ldots = 0$, as discussed previously. The plate motion is given by

$$h(x,t) = b_0\cos(\omega t - kx) \quad \text{for} \quad -1 < x < 1,$$

so that the trailing edge moves according to

$$h(1,t) = b_0\cos(\omega t - k).$$

The trailing edge will reach the highest position at $t = k/\omega$. At this instant the vortex sheet at the trailing edge, by (56) and (61), has strength

$$\gamma\left(1, \frac{k}{\omega}\right) = -\pi U Re\left[\frac{\lambda_0 + \lambda_1}{K_0(j\sigma) + K_1(j\sigma)} e^{j(k-\sigma)}\right].$$

Now for this case, by (66) and (68),

$$\beta_0 = 2b_0 J_0(k), \quad \beta_1 = -2b_0 jJ_1(k), \quad \gamma_0 = -2b_0 jkJ_0(k), \quad \gamma_1 = -2b_0 kJ_1(k).$$

Hence from (21*b*)

$$\lambda_0 + \lambda_1 = -[\gamma_0 + \gamma_1 + j\sigma(\beta_0 + \beta_1)] = -2b_0(\sigma - k)\,[J_1(k) + jJ_0(k)].$$

Furthermore,

$$K_0(j\sigma) + K_1(j\sigma) = -\tfrac{1}{2}\pi\{J_1(\sigma) + Y_0(\sigma) + j[J_0(\sigma) - Y_1(\sigma)]\},$$

where Y_n are the Bessel functions of the second kind. Hence

$$\gamma\left(1, \frac{k}{\omega}\right) = -4b_0 U \frac{(\sigma - k)\,H(\sigma, k)}{[J_1(\sigma) + Y_0(\sigma)]^2 + [J_0(\sigma) - Y_1(\sigma)]^2}, \tag{81}$$

where

$$H(\sigma, k) = \{J_1(k)\,[J_1(\sigma) + Y_0(\sigma)] + J_0(k)\,[J_0(\sigma) - Y_1(\sigma)]\}\cos(\sigma - k)$$
$$+ \{J_0(k)\,[J_1(\sigma) + Y_0(\sigma)] - J_1(k)\,[J_0(\sigma) - Y_1(\sigma)]\}\sin(\sigma - k).$$

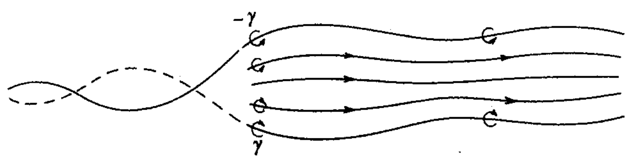

FIGURE 8. A qualitative sketch of the trailing vortex waves in the wake of a swimming plate.

By using the formula (Watson 1944, p. 77) $J_1(z)Y_0(z) - J_0(z)Y_1(z) = 2/\pi z$, it is readily seen that

$$H(k, k) = J_0^2(k) + J_1^2(k) + 2/\pi k,$$

which is positive definite for $0 < k < \infty$. Actually it can be shown that $H(\sigma, k) > 0$ for $0 < \sigma < \infty$, $0 < k < \infty$ (see Appendix). Therefore it follows that, so long as both σ and k are positive,

$$\gamma\left(1, \frac{k}{\omega}\right) \lesseqgtr 0 \quad \text{according as} \quad \sigma \gtreqless k.$$

Consider first the case $\sigma > k > 0$. When the trailing edge is at the highest position, the vorticity shed from the plate is negative, or in counterclockwise sense, as sketched in figure 8. On the other hand, when the trailing edge is at the lowest position, i.e. at the opposite phase in time, the vorticity shed from the plate is positive. This indicates that the more detailed structure of the wake will have the form of a series of parallel waves in which the vorticity varies across the wake from a negative strength at the top to a positive value at the bottom. If now the velocity field due to this vortex system is calculated, it is found that the fluid in the vortex wake is expelled downstream of the plate, moving in the form of a jet in the positive x-direction. Consequently, by the principle of action and reaction, the plate experiences a positive thrust, as the result shows. For the same reason, the thrust is negative if $0 < \sigma < k$.

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Appendix

This note is to show that $H(\sigma, k) > 0$ for $0 < \sigma < \infty$, $0 < k < \infty$, where (see (81))

$$H(\sigma,k) = [J_1(k)A(\sigma)+J_0(k)B(\sigma)]\cos(\sigma-k)+[J_0(k)A(\sigma)-J_1(k)B(\sigma)]\sin(\sigma-k),$$
$$A(\sigma) = J_1(\sigma)+Y_0(\sigma), \quad B(\sigma) = J_0(\sigma)-Y_1(\sigma).$$

By using Schafheitlin's integral representations for the Bessel functions (Watson 1944, p. 169), for $Re\, z > 0$,

$$J_0(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin(z+\frac{1}{2}\theta)}{\cos^{\frac{1}{2}}\theta\sin\theta} e^{-2z\cot\theta}\, d\theta,$$

$$Y_0(z) = -\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\cos(z+\frac{1}{2}\theta)}{\cos^{\frac{1}{2}}\theta\sin\theta} e^{-2z\cot\theta}\, d\theta,$$

and the relation $J_1(z) = -J_0'(z)$, $Y_1(z) = -Y_0'(z)$, one readily obtains

$$A(\sigma) = J_1(\sigma)+Y_0(\sigma) = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin(\sigma-\frac{1}{2}\theta)}{\cos^{\frac{1}{2}}\theta\sin^2\theta} e^{-2\sigma\cot\theta}\, d\theta,$$

$$B(\sigma) = J_0(\sigma)-Y_1(\sigma) = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\cos(\sigma-\frac{1}{2}\theta)}{\cos^{\frac{1}{2}}\theta\sin^2\theta} e^{-2\sigma\cot\theta}\, d\theta.$$

Using these results, one finds, after some straightforward manipulation,

$$H(\sigma,k) = \frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{[\cos\frac{1}{2}(\phi-\theta)-\cos\phi\cos\frac{1}{2}(4k+\phi-\theta)]}{(\cos\theta\cos\phi)^{\frac{1}{2}}\sin^2\theta\sin^2\phi} e^{-2(\sigma\cot\theta+k\cot\phi)}\, d\theta\, d\phi.$$

This shows that the sign of H depends only on the quantity in the square bracket in the numerator of the integrand, which is independent of σ, since the remaining part of the integrand is never negative. Therefore, the sign of $H(\sigma, k)$ is at least independent of σ. It follows that, for any given positive k, the function $H(\sigma, k)$ must have the same sign as the function

$$H_*(k) = J_0(k)\cos k + J_1(k)\sin k$$

which is the asymptotic representation of $\frac{1}{2}\pi\sigma H(\sigma, k)$ as $\sigma \to +0$.

Now the derivative of H_* is

$$H_*'(k) = -\frac{J_1(k)}{k}\sin k,$$

which vanishes at $k = m\pi$, $m = 0, 1, 2, \ldots$, and at $k = k_n$, the positive zeros of $J_1(k)$. The value of k_n is known to lie in $(n\pi + \pi/8, n\pi + \pi/4)$, $n = 1, 2, 3, \ldots$ (Watson 1944, p. 490). A study of the value of $H''_*(k)$ at these zeros of $H'_*(k)$ shows that the only minima of $H_*(k)$ are at $k = m\pi$, $m = 1, 2, 3, \ldots$. But

$$H_*(m\pi) = (-)^m J_0(m\pi) \quad (m = 1, 2, 3, \ldots),$$

which are all positive since it is known that $\operatorname{sgn}[J_0(m\pi)] = (-)^m$. Therefore $H_*(k) > 0$ for $0 < k < \infty$; the final result is then obvious.

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